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Lagrangian View of the Work-Rate Theorem

Julie J. Parish* and John E. Hurtado†
Texas A&M University,
College Station, Texas 77843-3141

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I. Introduction

THE work/energy-rate principle or, more simply, the work-rate theorem, is generally stated as "the rate of total energy is equal to the rate of change of work done by the generally nonconservative applied forces and constraint forces" [1]. One traditionally arrives at this statement using Newton's second law and the expression for the system energy [2]. Here, we first revisit this Newtonian view of the work-rate theorem and then instead consider the principle from a Lagrangian view.

The Lagrangian view is beneficial for several reasons. First, because the system kinetic and potential energies are used to construct the Lagrangian, only velocity-level kinematics are required. Second, expressions of Lagrange's equations exist for discrete, continuous, and hybrid systems and so the dynamics for a large class of systems can be described in the Lagrangian view. Third, the effect of ideal constraint forces, if any, are implicitly captured in Lagrangian mechanics in a straightforward way through the Lagrangian function.

Notationally, in lieu of summation symbols, Einstein's summation convention will be adopted [3].

II. Newtonian View of the Work-Rate Theorem

The traditional path to the work-rate theorem begins with the d'Alembert–Lagrange equation [1,4]:

$$(\boldsymbol{F} - m\ddot{\boldsymbol{x}}) \cdot \delta \boldsymbol{x} = 0 \tag{1}$$

This is an expression for a single-point-mass model for which the position vector and resultant force vector are given by $\boldsymbol{x}$ and $\boldsymbol{F}$. The position vector can be parameterized using a set of generalized coordinates and time: $\boldsymbol{x} = \boldsymbol{x}(q_1, q_2, q_3, \ldots, q_n, t)$. The explicit relation between the virtual displacement vector and the virtual displacement of the coordinates is then $\delta\boldsymbol{x} = (\partial\boldsymbol{x}/\partial q_i)\delta q_i$. This relationship can be used to rewrite the force vector, partitioned into potential, nonpotential, and ideal constraint forces:

$$\boldsymbol{F} \cdot \frac{\partial \boldsymbol{x}}{\partial q_i} = (\boldsymbol{f}_p + \boldsymbol{f}_n + \boldsymbol{f}_c) \cdot \frac{\partial \boldsymbol{x}}{\partial q_i} = P_i + Q_i = -\frac{\partial V}{\partial q_i} + Q_i \tag{2}$$

Here, P_i are forces derivable from a potential and Q_i are the generalized nonpotential forces. By definition, the ideal constraint forces do no virtual work, and so the resulting dot product is zero in this expression. The d'Alembert–Lagrange equation can now be restated:

$$\left(Q_i - \frac{\partial V}{\partial q_i} - m\ddot{\boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial q_i}\right)\delta q_i = 0 \tag{3}$$

Because the δq_i are independent and arbitrary, the parenthetical expression is identically zero. Further, when each term of the parenthetical expression is multiplied by its generalized velocity $\dot{q}_i$, the work-rate principle begins to emerge:

$$m\ddot{\boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial q_i}\dot{q}_i + \frac{\partial V}{\partial q_i}\dot{q}_i = Q_i \dot{q}_i \tag{4}$$

The left-hand side of this equation is related to the time derivative of the energy $E = T + V$, where $T = m(\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}})/2$ and $V = V(q)$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} + \frac{\partial V}{\partial q_i}\dot{q}_i = m\ddot{\boldsymbol{x}} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial q_i}\dot{q}_i + \frac{\partial \boldsymbol{x}}{\partial t}\right) + \frac{\partial V}{\partial q_i}\dot{q}_i \tag{5}$$

Together, Eqs. (4) and (5) reveal that not only is the change in system energy equal to the product of the generalized forces and velocities, as expected, but that explicit time dependence also contributes for rheonomic systems:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(T + V) = Q_i\dot{q}_i + m\ddot{\boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t} = Q_i\dot{q}_i + \boldsymbol{F} \cdot \frac{\partial \boldsymbol{x}}{\partial t} \tag{6}$$

If one were instead interested in the change in kinetic energy, the work rate simplifies to the traditional view:

$$\dot{T} = \boldsymbol{F} \cdot \dot{\boldsymbol{x}} \tag{7}$$

Note that Newton's second law must be employed to calculate $\boldsymbol{f}_c$, which is the ideal constraint force portion of the resultant force vector.

To illustrate the effect that the ideal constraint forces have on the work rate, consider the classic example of a point mass confined to move within a tube rotating in a plane in a gravity-free environment. The point mass has radial position $\boldsymbol{x} = r\hat{\boldsymbol{e}}_r$ with generalized coordinate r and mass ρ, and the tube rotates with prescribed angle $\theta = \omega t$ (ω constant), as shown in Fig. 1 [5].

The velocity and acceleration vectors of the system are $\dot{\boldsymbol{x}} = (\dot{r})\hat{\boldsymbol{e}}_r + (r\omega)\hat{\boldsymbol{e}}_\theta$ and $\ddot{\boldsymbol{x}} = (\ddot{r} - r\omega^2)\hat{\boldsymbol{e}}_r + (2\dot{r}\omega)\hat{\boldsymbol{e}}_\theta$, respectively. Using Newton's second law, the governing equations for the system are as follows:

$$F_r = 0 = \rho\ddot{r} - \rho r\omega^2; \qquad F_\theta = f_c = 2\rho\dot{r}\omega \tag{8}$$

The only force acting on the point mass is the ideal constraint force $\boldsymbol{f}_c$, and the energy rate can be found using Eq. (6):

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*Ph.D. Student, Department of Aerospace Engineering, TAMU-3141; julieparish@tamu.edu. Member AIAA.

†Associate Professor, Department of Aerospace Engineering, TAMU-3141; jehurtado@tamu.edu. Senior Member AIAA.

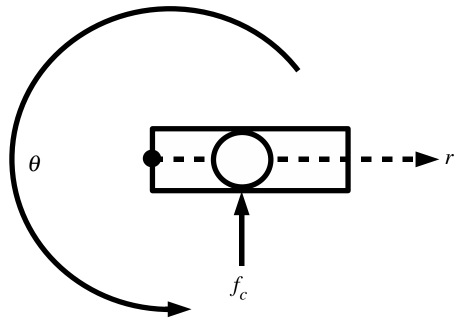

Fig. 1 Point mass in a rotating tube example.

$$\dot{E} = \boldsymbol{Q} \cdot \dot{\boldsymbol{q}} + \boldsymbol{F} \cdot \frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{f}_c \cdot \frac{\partial \boldsymbol{x}}{\partial t} = 2\rho r \dot{r} \omega^2 \tag{9}$$

Here, the explicit time dependence of the system is related to the prescribed motion of the tube or, equivalently, the $\hat{\boldsymbol{e}}_r$ direction. This example will also be visited from the Lagrangian perspective.

III. Lagrangian View of the Work-Rate Theorem: Discrete Systems

The work-rate theorem will now be investigated from the Lagrangian view, with Lagrange's equations for a discrete system as the starting point. The Lagrangian for a discrete system can be written as $L = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - V(\boldsymbol{q}, t)$, where T is the kinetic energy, V is the potential energy, $\boldsymbol{q}$ are the generalized coordinates, and $\dot{\boldsymbol{q}}$ are the generalized velocities. Lagrange's equations are then the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + P_i \tag{10}$$

The kinetic energy function for any finite-dimensional system can be partitioned into three terms: one term that is a quadratic function of the generalized velocities, another that is a linear function of the generalized velocities, and a third that is independent of the generalized velocities [6]:

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T_2 + T_1 + T_0 = \tfrac{1}{2} m_{ij} \dot{q}_i \dot{q}_j + n_i \dot{q}_i + o \tag{11}$$

In general, each coefficient is an explicit function of the generalized coordinates and time: $m_{ij} = m_{ij}(\boldsymbol{q}, t)$, $n_i = n_i(\boldsymbol{q}, t)$, and $o = o(\boldsymbol{q}, t)$. One can examine the explicit role that the coefficients m_{ij}, n_i, and o play in Lagrange's equations by taking the appropriate partial derivatives:

$$m_{ij}\ddot{q}_j + \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_j \dot{q}_k + \left(\frac{\partial m_{ij}}{\partial t}\right)\dot{q}_j + \frac{\partial n_i}{\partial t} - \frac{\partial o}{\partial q_i} = Q_i - \frac{\partial V}{\partial q_i} \tag{12}$$

As before, one of the first steps toward the work-rate principle is multiplying each term in the preceding set of equations by its generalized velocity $\dot{q}_i$ and summing over the entire collection:

$$m_{ij}\ddot{q}_j\dot{q}_i + \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_j\dot{q}_k\dot{q}_i + \left(\frac{\partial m_{ij}}{\partial t}\right)\dot{q}_j\dot{q}_i + \left(\frac{\partial n_i}{\partial t}\right)\dot{q}_i - \left(\frac{\partial o}{\partial q_i}\right)\dot{q}_i = Q_i\dot{q}_i - \left(\frac{\partial V}{\partial q_i}\right)\dot{q}_i \tag{13}$$

Separate from this, the time derivative of the kinetic energy can be computed:

$$\dot{T} = m_{ij}\ddot{q}_j\dot{q}_i + \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_j\dot{q}_k\dot{q}_i + \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial t}\right)\dot{q}_j\dot{q}_i + \left(\frac{\partial n_i}{\partial q_j}\right)\dot{q}_i\dot{q}_j + \left(\frac{\partial n_i}{\partial t}\right)\dot{q}_i + n_i\ddot{q}_i + \left(\frac{\partial o}{\partial q_i}\right)\dot{q}_i + \frac{\partial o}{\partial t} \tag{14}$$

Equations (13) and (14) can be used to find a relationship for the work rate of a general system:

$$\dot{T} + \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial t}\right)\dot{q}_j\dot{q}_i - \left(\frac{\partial n_i}{\partial q_j}\right)\dot{q}_i\dot{q}_j - n_i\ddot{q}_i - 2\left(\frac{\partial o}{\partial q_i}\right)\dot{q}_i - \frac{\partial o}{\partial t} = -\frac{\partial V}{\partial q_i}\dot{q}_i + Q_i\dot{q}_i \tag{15}$$

This equation is valid for systems that are rheonomic (which are explicitly time-dependent) and scleronomic (which are not). If the potential energy function is scleronomic, which is common, the work-rate theorem can be introduced:

$$\dot{E} = -\frac{1}{2}\left(\frac{\partial m_{ij}}{\partial t}\right)\dot{q}_j\dot{q}_i + \left(\frac{\partial n_i}{\partial q_j}\right)\dot{q}_i\dot{q}_j + n_i\ddot{q}_i + 2\left(\frac{\partial o}{\partial q_i}\right)\dot{q}_i + \frac{\partial o}{\partial t} + Q_i\dot{q}_i \tag{16}$$

This new result directly shows the change in energy as a result of both the rheonomic nature of the system and the contribution of nonpotential forces, unlike in the Newtonian view, which focuses on individual contributions of the forces. Furthermore, this view allows one to use two advantages of Lagrangian mechanics: neither acceleration-level kinematics nor ideal constraint forces must be computed. This second point is worth emphasizing. One critical advantage of Lagrangian mechanics is the elimination of ideal constraint forces. They never enter the picture. But before the introduction of Eq. (16), if one were interested in the change in energy of a system subject to ideal constraint forces, they would have to determine the ideal constraint forces even if a Lagrangian approach to dynamic modeling was used. Equation (16) shows that this is no longer necessary.

Again consider the example of the point mass in the rotating tube of Fig. 1. The kinetic energy for the system can be constructed from the velocity vector:

$$T = \tfrac{1}{2}\rho(\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}) = \tfrac{1}{2}\rho(\dot{r}^2 + r^2\omega^2) \tag{17}$$

Using the preceding developments, the following coefficients can be calculated:

$$m = \rho; \qquad n = 0; \qquad o = \tfrac{1}{2}\rho r^2 \omega^2 \tag{18}$$

The energy rate can then be constructed directly from the kinetic energy using Eq. (16):

$$\dot{E} = 2\frac{\partial o}{\partial q}\dot{q} = 2\frac{\partial o}{\partial r}\dot{r} = 2\rho r \dot{r}\omega^2 \tag{19}$$

Note that the constraint force did not have to be explicitly computed and that from the Lagrangian perspective one can essentially directly observe how the nonpotential forces and rheonomic components of the kinetic energy contribute to the energy rate.

A. Natural Systems

If the kinetic energy function is not explicitly a function of time, then the coefficients n_i and o are identically zero. Moreover, in this case, each coefficient m_{ij} is (at most) a function of the generalized coordinates only: $m_{ij} = m_{ij}(\boldsymbol{q})$. These systems are called *natural systems* and, assuming a schleronomic potential energy function, the work-rate expression takes a more simple form:

$$\dot{E} = Q_i\dot{q}_i \tag{20}$$

This result is consistent with the original result in Eq. (16) in that only one of the two contributing parts (i.e., the nonpotential contribution) influences the energy rate.

B. Constrained Systems

Occasionally, the generalized coordinates and velocities are not free, but are constrained by a set of functions that are linear in the $\dot{q}$:

$$c_{ji}\dot{q}_i + d_j = 0 \tag{21}$$

In general, the coefficients are an explicit function of the generalized coordinates and time: $c_{ji} = c_{ji}(\boldsymbol{q}, t)$ and $d_j = d_j(\boldsymbol{q}, t)$. When a system is subject to constraints, there are additional forces on the right side of Lagrange's equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + c_{ij}\lambda_j \tag{22}$$

The additional forces are constraint forces, which depend on the new (unknown) Lagrange multipliers λ_j. The presence of these additional forces slightly changes the general work-rate expression given by Eq. (16):

$$\dot{E} + \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial t}\right)\dot{q}_j\dot{q}_i - \left(\frac{\partial n_i}{\partial q_j}\right)\dot{q}_i\dot{q}_j - n_i\ddot{q}_i - 2\left(\frac{\partial o}{\partial q_i}\right)\dot{q}_i - \frac{\partial o}{\partial t} = Q_i\dot{q}_i - \lambda_j d_j \tag{23}$$

If the constraints are holonomic, that is, if the constraint functions in Eq. (22) are integrable, then the constraint equations can be reexamined:

$$\dot{\phi}_j(\boldsymbol{q}, t) = \left(\frac{\partial \phi_j}{\partial q_i}\right)\dot{q}_i + \frac{\partial \phi_j}{\partial t} = c_{ji}\dot{q}_i + d_j = 0 \tag{24}$$

If the holonomic constraint functions are scleronomic, then the coefficients d_j are identically zero [4]. Consequently, for constrained natural systems with a scleronomic potential energy function and subject to scleronomic holonomic constraints, one again finds that the work rate of the nonpotential forces equals the change in system energy:

$$\dot{E} = Q_i\dot{q}_i \tag{25}$$

As assumptions eliminate system functions or constraints that are explicitly time-dependent, the contributions to the work energy rate are found to only be from nonpotential forces acting on the system.

IV. Lagrangian View of the Work-Rate Theorem: Continuous Systems

The work-rate theorem can also be constructed for continuous systems via Lagrange's equations. Lee and Junkins [7] formulated a Lagrangian approach to construct the governing equations of motion for continuous dynamical systems. Here, the class of systems of interest are assumed to have only a single elastic body and therefore the Lagrangian can be written in the general form $L = L(w, \dot{w}, w', w'', x, t)$. Note that the overdot represents the operator $\mathrm{d}/\mathrm{d}t$ acting on the variable, whereas the prime represents the operator $\mathrm{d}/\mathrm{d}x$ acting on the variable. The Lagrangian is then constructed with the infinite-dimensional generalized coordinate $w(x, t)$, its derivatives, boundary terms L_B, and boundary conditions:

$$L = \int_{l_0}^{l} \hat{L}\mathrm{d}x + L_B; \qquad L_B = L_B(w(l), \dot{w}(l), w'(l), \dot{w}'(l), t)$$
$$\hat{L} = \hat{T} - \hat{V} = \hat{L}(w, \dot{w}, w', w'', x, t) \tag{26}$$

A hat over the Lagrangian indicates terms in the integrand of the Lagrangian. Lagrange's equations for a single-body continuous system are then the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{L}}{\partial \dot{w}}\right) - \frac{\partial \hat{L}}{\partial w} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{L}}{\partial w'}\right) - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \hat{L}}{\partial w''}\right) = \hat{f}_n \tag{27}$$

$$\left\{\frac{\partial \hat{L}}{\partial w'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{L}}{\partial w''}\right)\right\}\delta w\Big|_{l_0}^{l} + \left\{\frac{\partial L_B}{\partial w(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_B}{\partial \dot{w}(l)}\right)\right\}\delta w(l) + f_{n_1}\delta w(l) = 0 \tag{28}$$

$$\frac{\partial \hat{L}}{\partial w''}\delta w'\Big|_{l_0}^{l} + \left\{\frac{\partial L_B}{\partial w'(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_B}{\partial \dot{w}'(l)}\right)\right\}\delta w'(l) + f_{n_2}\delta w'(l) = 0 \tag{29}$$

Here, $\hat{f}_n$ is the nonpotential generalized force density vector related to w, and f_{n_1} and f_{n_2} are, respectively, the nonconservative force and torque vectors applied at the boundary. The governing equation of motion can instead be written in terms of the kinetic energy, rather than the Lagrangian:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{T}}{\partial \dot{w}}\right) - \frac{\partial \hat{T}}{\partial w} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{T}}{\partial w'}\right) - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \hat{T}}{\partial w''}\right) = \hat{f}_n + \hat{f}_p \tag{30}$$

Note that the potential generalized force density vector $\hat{f}_p$ accounts for the potential energy of the system:

$$\hat{f}_p = -\frac{\partial \hat{V}}{\partial w} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{V}}{\partial w'}\right) - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \hat{V}}{\partial w''}\right) \tag{31}$$

The kinetic energy can again be partitioned into terms that are quadratic, linear, and of no dependence on the generalized velocities $\dot{w}$:

$$\hat{T} = \tfrac{1}{2}m\dot{w}^2 + n\dot{w} + o \tag{32}$$

$$m = m(w, t); \qquad n = n(w, t); \qquad o = o(w, t) \tag{33}$$

Lagrange's equations can then be rewritten as the following:

$$m\ddot{w} + \frac{1}{2}\left(\frac{\partial m}{\partial w}\right)\dot{w} + \left(\frac{\partial m}{\partial t}\right)\dot{w} + \left(\frac{\partial n}{\partial t}\right) - \left(\frac{\partial o}{\partial w}\right) = \hat{f}_n + \hat{f}_p \tag{34}$$

As in the discrete case, this equation is post-multiplied by the generalized velocities:

$$m\ddot{w}\,\dot{w} + \frac{1}{2}\left(\frac{\partial m}{\partial w}\right)\dot{w}^2 + \left(\frac{\partial m}{\partial t}\right)\dot{w}^2 + \left(\frac{\partial n}{\partial t}\right)\dot{w} - \left(\frac{\partial o}{\partial w}\right)\dot{w} = \hat{f}_n\dot{w} + \hat{f}_p\dot{w} \tag{35}$$

Independently, one can construct the time rate of change of the kinetic energy:

$$\dot{\hat{T}} = m\ddot{w}\,\dot{w} + \frac{1}{2}\left(\frac{\partial m}{\partial w}\right)\dot{w}^3 + \frac{1}{2}\left(\frac{\partial m}{\partial t}\right)\dot{w}^2 + \left(\frac{\partial n}{\partial w}\right)\dot{w}^2 + \left(\frac{\partial n}{\partial t}\right)\dot{w} + n\ddot{w} + \left(\frac{\partial o}{\partial w}\right)\dot{w} + \frac{\partial o}{\partial t} \tag{36}$$

Equations (35) and (36) can be combined to produce the following equation:

$$\dot{\hat{T}} + \frac{1}{2}\left(\frac{\partial m}{\partial t}\right)\dot{w}^2 - \left(\frac{\partial n}{\partial w}\right)\dot{w}^2 - n\ddot{w} - 2\left(\frac{\partial o}{\partial w}\right)\dot{w} - \frac{\partial o}{\partial t} = \hat{f}_n\dot{w} + \hat{f}_p\dot{w} \tag{37}$$

Now consider the time rate of change of the potential energy:

$$\dot{\hat{V}} = \frac{\partial \hat{V}}{\partial w}\dot{w} + \frac{\partial \hat{V}}{\partial w'}\dot{w}' + \frac{\partial \hat{V}}{\partial w''}\dot{w}'' + \frac{\partial \hat{V}}{\partial t} \tag{38}$$

Again, let $\partial\hat{V}/\partial t = 0$. Integrating the simplified expression over the spatial domain via integration by parts, we have the following result:

$$\int_{l_0}^{l} \dot{\hat{V}}\,\mathrm{d}x = \int_{l_0}^{l}\left[\frac{\partial \hat{V}}{\partial w}\dot{w} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{V}}{\partial w'}\right)\dot{w} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \hat{V}}{\partial w''}\right)\dot{w}\right]\mathrm{d}x + \frac{\partial \hat{V}}{\partial w'}\dot{w}\bigg|_{l_0}^{l} + \frac{\partial \hat{V}}{\partial w''}\dot{w}'\bigg|_{l_0}^{l} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{V}}{\partial w''}\right)\dot{w}\bigg|_{l_0}^{l} \tag{39}$$

Using Eqs. (31), (37), and (39), the energy rate for continuous systems can be constructed:

$$\dot{E} = \int_{l_0}^{l}\dot{\hat{E}}\,\mathrm{d}x = \int_{l_0}^{l}(\dot{\hat{T}} + \dot{\hat{V}})\,\mathrm{d}x = \int_{l_0}^{l}\left[n_i\ddot{w} - \frac{1}{2}\left(\frac{\partial m}{\partial t}\right)\dot{w}^2 + \left(\frac{\partial n}{\partial w}\right)\dot{w}^2 + 2\left(\frac{\partial o}{\partial w}\right)\dot{w} + \frac{\partial o}{\partial t} + \hat{f}_n\dot{w}\right]\mathrm{d}x + \frac{\partial \hat{V}}{\partial w'}\dot{w}\bigg|_{l_0}^{l} + \frac{\partial \hat{V}}{\partial w''}\dot{w}'\bigg|_{l_0}^{l} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{V}}{\partial w''}\right)\dot{w}\bigg|_{l_0}^{l} \tag{40}$$

Note the presence of boundary conditions. Boundary forces can introduce energy to the system via these boundary conditions in conjunction with the original boundary conditions of Eqs. (28) and (29). That is, the original boundary conditions must first be considered to determine if the boundary conditions in Eq. (40) are nonzero. If no boundary forces exist, the energy rate can be simplified:.

$$\dot{E} = \int_{l_0}^{l}\left[n_i\ddot{w} - \frac{1}{2}\left(\frac{\partial m}{\partial t}\right)\dot{w}^2 + \left(\frac{\partial n}{\partial w}\right)\dot{w}^2 + 2\left(\frac{\partial o}{\partial w}\right)\dot{w} + \frac{\partial o}{\partial t} + \hat{f}_n\dot{w}\right]\mathrm{d}x \tag{41}$$

Note here that, just as in the discrete case, the time rate of change of the energy results from two contributions: one from the nonpotential forces and the other from the rheonomic nature of the system. Just as shown for the discrete case, if the system is scleronomic, the work energy rate will only be dependent on the contribution related to the nonpotential forces.

As an example, consider a cantilevered flexible body with mass density ρ, stiffness and inertia EI, and boundary force F, as shown in Fig. 2. The kinetic and potential energies for the system, along with the kinetic energy coefficients, can be written as follows:

$$\hat{T} = \tfrac{1}{2}\rho\dot{y}^2; \qquad \hat{V} = \tfrac{1}{2}EI(y'')^2 \tag{42}$$

$$m = \rho; \qquad n = 0; \qquad o = 0 \tag{43}$$

The geometric and natural spatial boundary conditions [Eqs. (28) and (29)] are the following:

$$y(0) = 0; \qquad y'(0) = 0; \qquad EIy''(l) = 0; \qquad EIy'''(l) = -F \tag{44}$$

These values can be used to then compute the nonzero energy-rate boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{V}}{\partial y''}\right)\dot{y}(l) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{V}}{\partial y''}\right)\dot{y}(0) = EIy'''(l)\dot{y}(l) = -F\dot{y}(l) \tag{45}$$

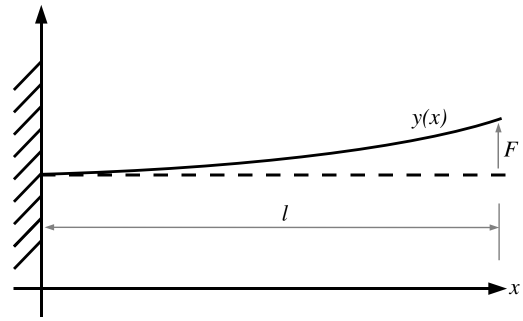

Fig. 2 Cantilevered flexible-body example.

Finally, the energy rate can be computed using Eq. (40):

$$\dot{E} = \int_{0}^{l}(0)\,\mathrm{d}x + \frac{\partial \hat{V}}{\partial y'}\dot{y}\bigg|_{0}^{l} + \frac{\partial \hat{V}}{\partial y''}\dot{y}'\bigg|_{0}^{l} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{V}}{\partial y''}\right)\dot{y}\bigg|_{0}^{l} = F\dot{y}(l) \tag{46}$$

Here, the nonpotential force is applied at the boundary $y(l)$, rather than distributed over the structure, and so the work energy rate is a function of velocity of that boundary and is introduced via the boundary conditions.

V. Conclusions

A Lagrangian perspective of the work-rate theorem demonstrates how one of the factors contributing to the time rate of change of the system energy can be viewed as arising from the rheonomic nature of a system as opposed to the traditional product of ideal constraint forces and velocities. The utility of the Lagrangian view is highlighted by features of Lagrangian mechanics, such as velocity-level kinematics and an inherent capture of the ideal constraint forces. Furthermore, because Lagrange's equations exist for several types of systems (discrete, continuous, constrained, etc.), this perspective can be easily extended to a large class of systems, many of which may acutely benefit from the probable reduced calculations generated from the Lagrangian, rather than Newtonian, view.

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